$\therefore \{S_n\}$ is bounded above

Now $S_{n+1} - S_n = u_1 + u_2 + u_3 + \cdots + u_n + u_{n+1} - u_1 - u_2 - u_3 - \cdots - u_n$

$= u_{n+1} \geq 0$

$\Rightarrow S_{n+1} - S_n \geq 0$

$\Rightarrow S_{n+1} > S_n$

$\therefore \{S_n\}$ is increasing sequence.

As we know that $\{S_n\}$ is increasing and bounded above

$\therefore \{S_n\}$ is convergent

$\therefore \{S_n\}$ is convergent $\Rightarrow \Sigma u_n$ is convergent.

Theorem :- If ΣU_n and ΣV_n be two series of posi ~~integer~~ terms such that

(a) There is a positive integer m and $K \in \mathbb{R}^+$ s.t.

$U_n \geq K V_n \ \forall \ n \geq m.$

(b) ΣV_n is divergent. then ΣU_n is divergent.

Proof :- If ΣU_n and ΣV_n be two series of positive terms such that

(a) There is a positive integer m and $K \in \mathbb{R}^+$ s

$U_n \geq K V_n \ \forall \ n \geq m.$

(b) ΣV_n is convergent.

Now we have to prove ΣU_n is convergent.

Let $S_n = U_1 + U_2 + U_3 + \text{------} + U_n$

$t_n = V_1 + V_2 + V_3 + \text{-------} + V_n$

$\forall n \geq m$, $S_n = \underbrace{U_1 + U_2 + \text{----} + U_{m-1}}_{a} + (U_m + U_{m+1} + \text{----} + U_n)$

$\geq a + kV_m + kV_{m+1} + \text{----} + kV_n$ $\because$ from (a)

$\geq a + k\left(t_n - (V_1 + V_2 + \text{---} + V_{m-1})\right)$

$\geq a + k(t_n - b)$ $\because$ from (b)

i.e., $\forall n \geq m$, $S_n \geq a + kt_n - kb \rightarrow (1)$

ΣV_n is divergent $\Rightarrow \{t_n\}$ is divergent.

$\Rightarrow$ for $G > 0$ $\exists$ $m_1 \in Z^+$ s.t $t_n > G$ $\forall n \geq m \rightarrow (2)$

Let $M = \max\{m, m_1\}$

Then (1) & (2) holds $\forall n \geq m$

$\therefore \forall n \geq m$, $S_n \geq a + kt_n - kb > a + kG - kb > G_1$

where $G_1 = a + kG - kb$

$\therefore \{S_n\}$ is convergent.

Hence ΣU_n is divergent.

* Test the convergence of $\sum_{n=1}^{\infty} \log\left(\frac{1}{n}\right)$.

Sol: Given that $\sum_{n=1}^{\infty} \log\left(\frac{1}{n}\right)$.

Let $\sum_{n=1}^{\infty} U_n = \sum_{n=1}^{\infty} \log(1/n)$

$= \sum_{n=1}^{\infty} (\log 1 - \log n)$

$= \sum_{n=1}^{\infty} (0 - \log n)$

$= - \sum_{n=1}^{\infty} \log n$

for $n \geq 1$, $n > \log n \Rightarrow -n < -\log n$

$\because \Sigma(-n)$ is divergent, by comparision test

$-\sum_{n=1}^{\infty} \log n = \sum_{n=1}^{\infty} U_n$ is divergent.

* Test the convergence of $\Sigma \frac{1}{2n^2+1}$.

Sol: let $U_n = \frac{1}{2n^2+1} > 0 \quad \forall n \in z^+$

$2n^2+1 > n^2 \Rightarrow \frac{1}{2n^2+1} < \frac{1}{n^2} \quad \forall n \in z^+$

Here $\Sigma V_n = \Sigma \frac{1}{n^2}$ is an auxilary series.

with $p = 2 > 1$

$\therefore \Sigma V_n = \Sigma \frac{1}{n^2}$ is convergent

$\therefore$ By comparision test $\Sigma U_n = \Sigma \frac{1}{2n^2+1}$ is convergent.

$\lim \frac{U_n}{V_n} \neq 0$

$\lim_{n\to\infty} \frac{\frac{1}{2n^2+1}}{1/n^2} \Rightarrow \lim_{n\to\infty} \frac{n^2}{2n^2+1}$

$\Rightarrow \lim_{n\to\infty} \frac{n^2}{n^2(2+1/n^2)}$

$\Rightarrow \lim_{n\to\infty} \frac{1}{2+1/n^2}$

$\Rightarrow \frac{1}{2} \neq 0$.

* Theorem :- Limit Comparision Test

Statement :- If ΣU_n and ΣV_n be two series of +ve terms and $\lim \frac{U_n}{V_n} = l \neq 0$, then ΣU_n and ΣV_n both converge or diverge

Proof :- let ΣU_n and ΣV_n be two series of +ve terms

i.e., $U_n > 0$, $V_n > 0 \quad \forall n \in z^+$

let $\lim_{n\to\infty} \frac{U_n}{V_n} = l \neq 0$

$\because \frac{U_n}{V_n} > 0 \quad \forall n \in z^+, \; l \geq 0$

let $\epsilon > 0$ be s.t $0 < \epsilon < l$. Then $l - \epsilon > 0$ & $l + \epsilon > 0$

then $\lim \frac{U_n}{V_n} = l \Rightarrow$ for $\epsilon > 0 \; \exists m \in z^+$ s.t $\left|\frac{U_n}{V_n} - l\right| < \epsilon \quad \forall n \geq m$

$\Rightarrow -\epsilon < \frac{U_n}{V_n} - l < \epsilon \quad \forall n \geq m$

$\Rightarrow l - \epsilon < \frac{U_n}{V_n} < l + \epsilon \quad \forall n \geq m$

$\Rightarrow (l-\epsilon) V_n < U_n < (l+\epsilon) V_n \quad \forall n \geq m$

$\Rightarrow K_1 V_n < U_n < K_2 V_n \quad \forall n \geq m$

where $K_1 = l - \epsilon > 0$, $K_2 = l + \epsilon > 0$.

Case i:- let ΣU_n be convergent.

Now Consider $K_1 V_n < U_n \quad \forall n \geq m$

$\Rightarrow V_n < \frac{1}{K} U_n \quad \forall n \geq m$

Now by the theorem " If ΣU_n and ΣV_n be two series of +ve terms st.

(a) $\exists$ a positive integer m & $k \in \mathbb{R}^+$ s.t $U_n \le k \vartheta_n$ $\forall n \ge m$

(b) $\Sigma \vartheta_n$ is convergent, then ΣU_n is convergent.

$\therefore \Sigma \vartheta_n$ is convergent.

Case ii:- Let $\Sigma \vartheta_n$ be convergent

Consider $U_n < K_2 \vartheta_n$ $\forall n \ge m$

Then by above theorem ΣU_n is convergent.

Case iii:- Let ΣU_n be divergent

Consider $U_n < K_2 \vartheta_n$ $\forall n \ge m$

By the theorem " If ΣU_n and $\Sigma \vartheta_n$ be two series of +ve terms s.t

(a) $\exists$ +ve integer m and $k \in \mathbb{R}^+$ s.t $U_n > k \vartheta_n$ $\forall n \ge m$

(b) $\Sigma \vartheta_n$ is divergent ~~convergent~~ then ΣU_n is divergent

$\therefore \Sigma \vartheta_n$ is divergent.

Case iv:- Let $\Sigma \vartheta_n$ be divergent.

Consider $K_1 \vartheta_n < U_n$ $\forall n \ge m$ i.e, $U_n > K_1 \vartheta_n$ $\forall n \ge m$

By above theorem ΣU_n is divergent.

Hence ΣU_n & $\Sigma \vartheta_n$ both converge or diverge together

* Test the convergence of series $\frac{1\cdot 2}{3\cdot 4\cdot 5} + \frac{2\cdot 3}{4\cdot 5\cdot 6} + \frac{3\cdot 4}{5\cdot 6\cdot 7} + \cdots\cdots$

Sol: Given Series $\frac{1\cdot 2}{3\cdot 4\cdot 5} + \frac{2\cdot 3}{4\cdot 5\cdot 6} + \frac{3\cdot 4}{5\cdot 6\cdot 7} + \cdots\cdots$

Here $U_n = \frac{n(n+1)}{(n+2)(n+3)(n+4)} > 0$ $\forall n \in \mathbb{Z}^+$

$\therefore \Sigma U_n$ is a series of +ve terms.

Take $V_n = \frac{n^2}{n^3} = \frac{1}{n} > 0$ $\forall n \in \mathbb{Z}^+$

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{\frac{n(n+1)}{(n+2)(n+3)(n+4)}}{1/n}$$

$$= \lim_{n\to\infty} \frac{n^2 \cdot n(1+1/n)}{n^3(1+2/n)(1+3/n)(1+4/n)}$$

$= 1 \neq 0$.

Now by the limit comparision test ΣU_n & ΣV_n both converge or diverge together.

$\Sigma \vartheta_n = \Sigma \frac{1}{n}$ is an auxilary series with $p=1$

$\therefore \Sigma V_n$ is divergent

Hence ΣU_n is divergent.

* Test the convergence of $\Sigma\left(\sqrt{n^4+1} - n^2\right)$

Sol: Let $U_n = \sqrt{n^4+1} - n^2 \times \frac{\sqrt{n^4+1} + n^2}{\sqrt{n^4+1} + n^2}$

$$= \frac{\cancel{n^4}+1 - \cancel{n^4}}{\sqrt{n^4+1} + n^2}$$

$$= \frac{1}{\sqrt{n^4+1} + n^2} > 0 \quad \forall n \in \mathbb{Z}^+$$

Take $v_n = \frac{1}{n^2} > 0 \ \forall n \in z^+$

$\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{\frac{1}{n^2\left(\sqrt{1+\frac{1}{n^4}}+1\right)}}{\frac{1}{n^2}}$

$= \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^4}}+1}$

$= \frac{1}{2} \neq 0$

By limit comparision test Σu_n & Σv_n both converge (or) diverge together

Here $\Sigma v_n = \Sigma \frac{1}{n^2}$ is an auxilary series with $p = 2 > 1$

$\therefore \Sigma v_n$ is convergent.

Hence Σu_n is convergent

H.W

* Test the convergence of following series.

(i) $\sum_{n=1}^{\infty} \left(\sqrt{n+1} - \sqrt{n}\right)$ (ii) $\sum_{n=1}^{\infty} \left(\sqrt{n^3+1} - \sqrt{n^3}\right)$

(iii) $\sum_{n=1}^{\infty} \left(\sqrt{n^2+1} - n\right)$ (iv) $\sum_{n=1}^{\infty} \left(\sqrt{n^4+1} - \sqrt{n^4-1}\right)$

Sol: (i) let $u_n = \sqrt{n+1} - \sqrt{n} \times \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$

$= \frac{\cancel{n}+1-\cancel{n}}{\sqrt{n+1}+\sqrt{n}}$

$= \frac{1}{\sqrt{n+1}+\sqrt{n}} > 0 \ \forall n \in z^+$

Take $v_n \Rightarrow u_n = \frac{1}{\sqrt{n}\left[\sqrt{1+\frac{1}{n}}+1\right]} > 0 \ \forall n \in z^+$

Take $v_n = \frac{1}{\sqrt{n}} > 0 \ \forall n \in z^+$

$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}\left[\sqrt{1+\frac{1}{n}}+1\right]}$

$= \frac{1}{2} \neq 0$

By limit comparision test both converge (or) diverges together

Here $\Sigma v_n = \Sigma \frac{1}{\sqrt{n}} = \frac{1}{n^{1/2}}$ is an auxilary series with $p = \frac{1}{2} < 1$ then it is divergent

$\therefore \Sigma v_n$ is divergent

Σu_n is divergent

(ii) let $u_n = \sqrt{n^3+1} - \sqrt{n^3} \times \frac{\sqrt{n^3+1}+\sqrt{n^3}}{\sqrt{n^3+1}+\sqrt{n^3}}$

$= \frac{\cancel{n^3}+1-\cancel{n^3}}{\sqrt{n^3+1}+\sqrt{n^3}}$

$= \frac{1}{\sqrt{n^3+1}+\sqrt{n^3}} > 0 \ \forall n \in z^+$

$v_n = \frac{1}{\sqrt{n^3}}$ i.e, $\Sigma v_n = \Sigma \frac{1}{\sqrt{n^3}}$ is Auxilary series

$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^3+1}+\sqrt{n^3}}}{\frac{1}{\sqrt{n^3}}}$

$= \lim_{n\to\infty} \frac{\sqrt{n^3}}{\sqrt{n^3+1}+\sqrt{n^3}}$

$$= \lim_{n\to\infty} \frac{\sqrt{n^3}}{\sqrt{n^3}\left[\sqrt{1+\frac{1}{n^3}}+1\right]}$$

$$\therefore \lim_{n\to\infty} \frac{1}{1+1} = \frac{1}{2} \neq 0$$

By limit comparision test Σu_n, Σv_n both converge & diverge together.

Here $\Sigma v_n = \Sigma \frac{1}{\sqrt{n^3}} = \frac{1}{n^{3/2}}$ is an auxilary series with $P = \frac{3}{2} > 1$.

$\therefore \Sigma v_n$ is convergent

Hence Σu_n is convergent.

(ii) let $u_n = \sqrt{n^2+1} - n$

$$= \frac{\left[\sqrt{n^2+1}-n\right]\left[\sqrt{n^2+1}+n\right]}{\sqrt{n^2+1}+n}$$

$$= \frac{n^2+1-n^2}{\sqrt{n^2+1}+n} = \frac{1}{n\sqrt{1+\frac{1}{n^2}}+n} = \frac{1}{n\left[1+\frac{1}{\sqrt{n^2}}+1\right]} > 0 \ \forall n \in$$

Take $v_n = \frac{1}{n} > 0 \ \forall n \in z^+$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{n\left[1+\frac{1}{\sqrt{n^2}}+1\right]} \cdot \frac{n}{1} = \frac{1}{1+1} = \frac{1}{2} \neq 0$$

By limit comparision test Σu_n, Σv_n both converge & diverge together

Here $\Sigma v_n = \Sigma \frac{1}{n}$ is an auxilary series with $p = 1 \leq 1$

$\therefore \Sigma v_n$ is divergent

$\therefore \Sigma u_n$ is divergent.

(iv) let $u_n = \sqrt{n^4+1} - \sqrt{n^4-1}$

$$= \frac{\left[\sqrt{n^4+1}-\sqrt{n^4-1}\right]\left[\sqrt{n^4+1}+\sqrt{n^4-1}\right]}{\sqrt{n^4+1}+\sqrt{n^4-1}}$$

$$= \frac{n^4+1-n^4+1}{\sqrt{n^4+1}+\sqrt{n^4-1}} = \frac{2}{\sqrt{n^4+1}+\sqrt{n^4-1}}$$

$$= \frac{2}{n^2\left[\sqrt{1+\frac{1}{n^4}}+\sqrt{1-\frac{1}{n^4}}\right]} > 0 \ \forall n \in$$

Take $v_n = \frac{1}{n^2} > 0 \ \forall n \in z^+$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{2}{n^2\left[\sqrt{1+\frac{1}{n^4}}+\sqrt{1-\frac{1}{n^4}}\right]} \times n^2$$

$$\lim_{n\to\infty} \frac{2}{\sqrt{1+\frac{1}{n^4}}+\sqrt{1-\frac{1}{n^4}}} = \frac{2}{2} = 1 \neq 0$$

By limit comparision test Σu_n, Σv_n both converges & diverges together.

Here $\Sigma v_n = \Sigma \frac{1}{n^2}$ is an auxilary series with $p = 2 > 1$

$\therefore \Sigma v_n$ is convergent

$\therefore \Sigma u_n$ is ~~divergent~~ convergent

* Test the convergence of $\sum \frac{1}{2^n+3^n}$

Sol: Given that $u_n = \frac{1}{2^n+3^n} = \frac{1}{3^n\left(\left(\frac{2}{3}\right)^n+1\right)} > 0 \ \forall\ n \in z^+$

Take $v_n = \frac{1}{3^n} > 0 \ \forall\ n \in z^+$

$$\lim \frac{u_n}{v_n} = \lim \frac{\frac{1}{3^n\left(\left(\frac{2}{3}\right)^n+1\right)}}{1/3^n} = \frac{1}{0+1} = 1 \neq 0$$

$\therefore$ By limit comparision test $\sum u_n, \sum v_n$ both converge & diverge together.

$\sum v_n = \sum \frac{1}{3^n} = \sum \left(\frac{1}{3}\right)^n$ is a geometric series with $r = \frac{1}{3} < 1$

$\therefore \sum v_n$ is convergent

Hence $\sum u_n$ is convergent.

* Test the convergence of $\sum_{n=1}^{\infty} \left(\sqrt[3]{n^3+1} - n\right)$

Sol: Given that $\sum_{n=1}^{\infty} \sqrt[3]{n^3+1} - n$

Here $u_n = \sqrt[3]{n^3+1} - n$ $\qquad a^3-b^3 = (a-b)(a^2+ab+b^2)$

$$= (n^3+1)^{1/3} - (n^3)^{1/3}$$

$$= \frac{\left[(n^3+1)^{1/3} - (n^3)^{1/3}\right]\left[(n^3+1)^{2/3} + (n^3+1)^{1/3}(n^3)^{1/3} + (n^3)^{2/3}\right]}{(n^3+1)^{2/3} + (n^3+1)^{1/3}(n^3)^{1/3} + (n^3)^{2/3}}$$

$$= \frac{\left[(n^3+1)^{1/3}\right]^3 - \left[(n^3)^{1/3}\right]^3}{(n^3+1)^{2/3} + (n^3+1)^{1/3} n + n^2}$$

$$= \frac{\cancel{n^3}+1-\cancel{n^3}}{(n^3+1)^{2/3} + (n^3+1)^{1/3} n + n^2}$$

$$= \frac{1}{(n^3)^{2/3}\left[1+\frac{1}{n^3}\right]^{2/3} + (n^3)^{1/3}\left(1+\frac{1}{n^3}\right)^{1/3} n + n^2}$$

$$= \frac{1}{n^2\left[\left(1+\frac{1}{n^3}\right)^{2/3} + \left[1+\frac{1}{n^3}\right]^{1/3} + 1\right]} > 0 \ \forall\ n \in z^+$$

Take $v_n = \frac{1}{n^2} > 0 \ \forall\ n \in z^+$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\frac{1}{\cancel{n^2}\left[\left(1+\frac{1}{n^3}\right)^{2/3} + \left(1+\frac{1}{n^3}\right)^{1/3} + 1\right]}}{1/\cancel{n^2}}$$

$$= \frac{1}{\left[1+\frac{1}{n^3}\right]^{2/3} + \left[1+\frac{1}{n^3}\right]^{1/3} + 1}$$

$$= \frac{1}{1+1+1} = \frac{1}{3} \neq 0$$

$\therefore$ By limit comparision test $\sum u_n, \sum v_n$ both converge & diverge together.

Here $\sum v_n = \sum \frac{1}{n^2}$ is an auxilarity series with $p = 2 > 1$

$\therefore \sum v_n$ is convergent

Hence $\sum u_n$ is convergent.

* Test the convergence of $\sum \frac{n^p}{(1+n)^q}$

Sol: Given that $u_n = \frac{n^p}{(1+n)^q} > 0 \ \forall\ n \in z^+$

Take $V_n = \frac{n^p}{n^q} = \frac{1}{n^{q-p}} > 0 \ \forall n \in Z^+$

$\lim \frac{U_n}{V_n} = \lim \frac{n^p}{(1+n)^q} \times \frac{n^{q-p}}{1}$

$= \lim_{n\to\infty} \frac{n^{p+q-p}}{n^q(1+\frac{1}{n})^q} = \lim_{n\to\infty} \frac{n^q}{n^q(1+\frac{1}{n})^q}$

$= 1 \neq 0$

$\therefore$ By limit comparision test ΣU_n, ΣV_n both converge & diverge together.

Here $\Sigma V_n = \Sigma \frac{1}{n^{q-p}}$ is an auxilary series

$\therefore \Sigma V_n$ converges if $q-p>1$ i.e., $q>1+p$

ΣV_n diverges if $q-p\leq 1$ i.e., $q\leq 1+p$

Hence ΣU_n Converges if $q>1+p$ &

diverges if $q\leq 1+p$

Comparision Test of the Second Type:

Theorem: let ΣU_n & ΣV_n be two series of positive terms if $\exists\, m \in Z^+ \ni \frac{U_{n+1}}{U_n} \leq \frac{V_{n+1}}{V_n} \ \forall n \geq m$

then (i) ΣU_n converges if ΣV_n converges.

(ii) ΣV_n diverges if ΣU_n diverges.

proof: Let ΣU_n & ΣV_n be two series of positive terms.

Given that $\exists\, m \in z^+ \ni \frac{U_{n+1}}{U_n} \leq \frac{V_{n+1}}{V_n} \ \forall n \geq m$ i.e,

$\forall n \geq m \quad \frac{U_{n+1}}{V_{n+1}} \leq \frac{U_n}{V_n}$

putting $n = m, m+1, \ldots n-1$ we haget

$\frac{U_{m+1}}{V_{m+1}} \leq \frac{U_m}{V_m}$

$\frac{U_{m+2}}{V_{m+2}} \leq \frac{U_{m+1}}{V_{m+1}}$

$\frac{U_{m+3}}{V_{m+3}} \leq \frac{U_{m+2}}{V_{m+2}}$,

$\vdots$

$\frac{U_n}{V_n} \leq \frac{U_{n-1}}{V_{n-1}}$

Combining these inequalities we have

$\frac{U_m}{V_m} \geq \frac{U_{m+1}}{V_{m+1}} \geq \frac{U_{m+2}}{V_{m+2}} \geq \frac{U_{m+3}}{V_{m+3}} \geq \cdots \geq \frac{U_{n-1}}{V_{n-1}} \geq \frac{U_n}{V_n}$

$\Rightarrow \frac{U_m}{V_m} \geq \frac{U_n}{V_n}$

$\Rightarrow \left(\frac{U_m}{V_m}\right) V_n > U_n \Rightarrow U_n \leq K V_n$, where $K = \frac{U_m}{V_m} \in R^+$

$\therefore$ If ΣV_n converges then by comparision test ΣU_n converges

If ΣU_n diverges then by comparision test ΣV_n diverges. $\quad [\because V_n \geq \frac{1}{K} U_n \ \forall n \in z^+]$

* If Σu_n is a series of positive terms and Σu_n is convergent then $\Sigma \frac{u_n}{1+u_n}$ is convergent.

Sol: let Σu_n be a series of positive terms and Σu_n is convergent i.e., $u_n > 0 \ \forall n \in Z^+ \Rightarrow 1+u_n > 1 \ \forall n \in Z^+$

$$\Rightarrow \frac{1}{1+u_n} < 1 \ \forall n \in Z^+$$

$$\Rightarrow \frac{u_n}{1+u_n} < u_n \ \forall n \in Z^+$$

$\therefore$ By comparision test Σu_n is convergent

$\Rightarrow \Sigma \frac{u_n}{1+u_n}$ is convergent.

* Test the convergence of $\sum_{n=1}^{\infty} \frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots(2n)}$

Sol:
$$u_n = \frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots(2n)}$$

$$u_{n+1} = \frac{1\cdot3\cdot5\cdots(2n-1)(2(n+1)-1)}{2\cdot4\cdot6\cdots(2n)(2(n+1))}$$

$$= \frac{1\cdot3\cdot5\cdots(2n-1)(2n+1)}{2\cdot4\cdot6\cdots(2n)(2(n+1))}$$

$$= \frac{1\cdot3\cdot5\cdots(2n-1)(2n+1)}{2\cdot4\cdot6\cdots(2n)(2n+2)}$$

$$\frac{u_{n+1}}{u_n} = \frac{\cancel{1\cdot3\cdot5\cdots(2n-1)}(2n+1)}{\cancel{2\cdot4\cdot6\cdots(2n)}(2n+2)} \times \frac{\cancel{2\cdot4\cdot6\cdots(2n)}}{\cancel{1\cdot3\cdot5\cdots(2n-1)}}$$

$$= \frac{2n+1}{2n+2}$$

Take $v_n = 1/n$

Then $v_{n+1} = \frac{1}{n+1}$

$$\frac{v_{n+1}}{v_n} = \frac{1}{n+1} \times n = \frac{n}{n+1} = \frac{2n}{2n+2}$$

Then $\frac{u_{n+1}}{u_n} > \frac{v_{n+1}}{v_n} \ \forall n \in Z^+$

$\Sigma v_n = \Sigma \frac{1}{n}$ is an auxilary series with $p=1$

$\therefore v_n$ is divergent.

Imp

* Cauchy's n^{th} root test:

Statement: If Σu_n is a series of positive terms such that $\lim_{n\to\infty} u_n^{1/n} = l$ then

(a) Σu_n converges if $l < 1$ and

(b) Σu_n diverges if $l > 1$

(c) The test fails to decide nature of series if $l = 1$

proof: let Σu_n be series of positive terms such that $\lim_{n\to\infty} u_n^{1/n} = l$

Since $u_n > 0 \ \forall n \in Z^+$ and $u_n^{1/n}$ stands positively n^{th} roots of u_n $\lim_{n\to\infty} u_n^{1/n} = l \geq 0$

$\lim u_n^{1/n} = l \Rightarrow$ for $\varepsilon > 0$ $\exists\, m \in Z^+ \ni |u_n^{1/n} - l| < \varepsilon$ $\forall\, n \geq m$

$\Rightarrow l - \varepsilon < u_n^{1/n} < l + \varepsilon \ \forall\, n \geq m$

$\Rightarrow (l-\varepsilon)^n < u_n < (l+\varepsilon)^n \ \forall\, n \geq m$ — (1)

Case(i): let $l < 1$

choose $\varepsilon > 0$ suchthat $k = l + \varepsilon < 1$ then

$0 \leq l < k < 1$

from (1), we have

$u_n < (l+\varepsilon)^n \ \forall\, n \geq m$

$u_n < k^n \ \forall\, n \geq m$

$\therefore \Sigma k^n$ is a geometric series with common ratio

$k < 1$, $\therefore \Sigma k^n$ is convergent.

Hence by comparision test Σu_n is convergent

Case(ii): let $l > 1$

choose $\varepsilon > 0 \ \ni\ k = l - \varepsilon > 1$

from(1) $(l-\varepsilon)^n < u_n \ \forall\, n \geq m$

$\Rightarrow u_n > k^n \ \forall\, n \geq m$

$\therefore \Sigma k^n$ is divergent

Hence by camparision test Σu_n is divergent.

Case(iii): let $l = 1$

consider a series $\Sigma u_n = \Sigma 1/n$

Here $\lim\limits_{n\to\infty} u_n^{1/n} = \lim\limits_{n\to\infty} (1/n)^{1/n} = \lim\limits_{n\to\infty} \dfrac{1}{\sqrt[n]{n}}$

$\Sigma \frac{1}{n}$ is an auxilary series with $p = 1$

$\therefore \Sigma\, 1/n$ is divergent

consider a series $\Sigma u_n = \Sigma \frac{1}{n^2}$ $\quad \because \sqrt[n]{n} = 1$

$\lim\limits_{n\to\infty} u_n^{1/n} = \lim\limits_{n\to\infty} \left(\frac{1}{n^2}\right)^{1/n} = \lim\limits_{n\to\infty} \left(\frac{1}{n^{1/n}}\right)^2 = 1$

$\Sigma \frac{1}{n^2}$ is an auxilary series with $p = 2 > 1$

$\therefore \Sigma \frac{1}{n^2}$ is convergent

Hence the test fails to decide the nature of series if $l = 1$.

* Test for convergence $\Sigma \dfrac{2^n}{n^3}$

Sol: let $u_n = \dfrac{2^n}{n^3} > 0 \ \forall\, n \in Z^+$

$\lim\limits_{n\to\infty} u_n^{1/n} = \lim\limits_{n\to\infty} \left[\dfrac{2^n}{n^3}\right]^{1/n} = \lim\limits_{n\to\infty} \dfrac{2}{(n^{1/n})^3}$

$= \lim\limits_{n\to\infty} \dfrac{2}{1} = 2 > 1$

$\therefore$ By cauchy's n^{th} root test $\Sigma \dfrac{2^n}{n^3}$ is divergent.

* Test for convergence $\Sigma \left(1 + \frac{1}{n}\right)^{-n^2}$

Sol: let $u_n = \left(1 + \frac{1}{n}\right)^{-n^2} > 0 \ \forall\, n \in Z^+$

$\lim\limits_{n\to\infty} u_n^{1/n} = \lim\limits_{n\to\infty} \left[\left(1 + \frac{1}{n}\right)^{-n^2}\right]^{1/n}$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{-n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n}$$
$$= \frac{1}{e} < 1$$

$\therefore$ By cauchy's n^{th} root test $\Sigma\left(1+\frac{1}{n}\right)^{-n^2}$ is convergent

* Test for convergence $\sum_{n=1}^{\infty} \frac{1}{n^3}\left[\frac{n+2}{n+3}\right]^n x^n \ \forall x > 0.$

Sol: let $u_n = \frac{1}{n^3}\left[\frac{n+2}{n+3}\right]^n x^n > 0 \ \forall n \in z^+$

$$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty}\left(\frac{1}{n^3}\left[\frac{n+2}{n+3}\right]^n \cdot x^n\right)^{1/n}$$
$$= \lim_{n\to\infty} \left[\frac{1}{n^3}\right]^{1/n} \cdot \left(\left[\frac{n+2}{n+3}\right]^n\right)^{1/n} \cdot \left[x^n\right]^{1/n}$$
$$= \lim_{n\to\infty} \frac{1}{(n^{1/n})^3} \left(\frac{n+2}{n+3}\right) x$$
$$= \lim_{n\to\infty} \frac{1}{(n^{1/n})^3} \frac{\cancel{x}(1+2/n)}{\cancel{x}(1+3/n)} \cdot x$$
$$= 1 \cdot 1 \cdot x$$
$$= x$$

$\therefore$ By cauchy's n^{th} root test Σu_n converges if $x<1$
diverges if $x>1$

If $x=1$ then the series becomes
$$\sum_{n=1}^{\infty} \frac{1}{n^3}\left(\frac{n+2}{n+3}\right)^n$$

Here $u_n = \frac{1}{n^3}\left(\frac{n+2}{n+3}\right)^n > 0 \ \forall n \in z^+$

let $v_n = \frac{1}{n^3}$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{\cancel{n^3}}\left(\frac{n+2}{n+3}\right)^n \times \frac{\cancel{n^3}}{1}$$
$$= \lim_{n\to\infty} \left(\frac{n+2}{n+3}\right)^n$$
$$= \lim_{n\to\infty} \frac{\cancel{n^n}(1+2/n)^n}{\cancel{n^n}(1+3/n)^n}$$
$$= \frac{e^2}{e^3} = \frac{1}{e} \neq 0$$

$\therefore$ By limit comparision test Σu_n & Σv_n both converge or diverge together.

$\Sigma v_n = \Sigma \frac{1}{n^3}$ is an auxilary series with $p = 3 > 1$

$\therefore \Sigma v_n$ is convergent.

Hence Σu_n is convergent.

$\therefore \sum_{n=1}^{\infty} \frac{1}{n^3}\left[\frac{n+2}{n+3}\right]^n x^n \ \forall x>0$ converges if $x \leq 1$ &
diverges if $x>1$

* Test for convergence $\frac{2}{1^2}x + \frac{3^2}{2^3}x^2 + \dots + \frac{(n+1)^n}{n^{n+1}}x^n + \dots$ $(x>0)$

Sol: The given series is $\frac{2}{1^2}x + \frac{3^2}{2^3}x^2 + \dots + \frac{(n+1)^n}{n^{n+1}}x^n + \dots$
$$= \sum_{n=1}^{\infty} \frac{(n+1)^n}{n^{n+1}} x^n$$

let $u_n = \frac{(n+1)^n}{n^{n+1}} x^n > 0 \ \forall n \in z^+$

$$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} \left[\frac{(n+1)^n}{n^n \cdot n} x^n\right]^{1/n}$$

$$= \lim_{n\to\infty} \frac{n+1}{n \cdot n^{1/n}} \cdot x$$

$$= \lim_{n\to\infty} \frac{1}{n^{1/n}} \left[1+\frac{1}{n}\right] x$$

$$= 1 \cdot 1 \cdot x = x$$

$\therefore$ By cauchy's n^{th} root test Σu_n converges if $x<1$, diverges if $x>1$

If $x=1$ then the series becomes

$$\sum_{n=1}^{\infty} \frac{(n+1)^n}{n^{n+1}}$$

Here $u_n = \dfrac{(n+1)^n}{n^n \cdot n} > 0 \ \forall n \in Z^+$

$$= \left(1+\frac{1}{n}\right)^n \cdot \frac{1}{n} > 0 \ \forall n \in Z^+$$

Take $v_n = \dfrac{1}{n}$

Now $\displaystyle\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n \cdot \frac{1}{\cancel{n}} \times \cancel{n}$

$$= \left(1+\frac{1}{n}\right)^n = e \neq 0$$

$\therefore$ By limit comparision test Σu_n & Σv_n converge & diverges together

$\Sigma v_n = \Sigma \frac{1}{n}$ is an auxilary series with $p=1$

$\therefore \Sigma v_n$ is divergent.

$\therefore \Sigma u_n$ is divergent.

$\therefore \Sigma u_n = \Sigma \dfrac{(n+1)^n}{n^{n+1}} \cdot x^n > 0$ converges if $x<1$ & diverges if $x \geq 1$

* Test for convergence $\frac{1}{3} + \left(\frac{2}{5}\right)^2 + \left(\frac{3}{7}\right)^3 + \cdots + \left(\frac{n}{2n+1}\right)^n + \cdots$

* Test for convergence $\Sigma \dfrac{x^n}{n^n}$, $x>0$

* Test for convergence $\Sigma \left(\dfrac{n+1}{2n+5}\right)^n$

* Test for convergence $\Sigma \dfrac{n^2}{(n+1)^{n^2}}$

* Test for convergence $\Sigma n \cdot e^{-n^2}$

Sol: The given series $\frac{1}{3} + \left(\frac{2}{5}\right)^2 + \left(\frac{3}{7}\right)^3 + \cdots + \left(\frac{n}{2n+1}\right)^n + \cdots$

$$= \sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^n$$

let $u_n = \left(\dfrac{n}{2n+1}\right)^n > 0 \ \forall n \in Z^+$

$$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} \left[\left(\frac{n}{2n+1}\right)^n\right]^{1/n} = \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}(2+\frac{1}{n})}$$

$$= \lim_{n\to\infty} \frac{1}{2+1/n} = \frac{1}{2} < 1$$

$\therefore$ By cauchy's n^{th} root test Σu_n is convergent.

2 Sol: let $u_n = \frac{x^n}{n^n} > 0 \; \forall n \in z^+$

$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} \left[\frac{x^n}{n^n}\right]^{1/n} = \lim_{n\to\infty} \frac{x}{(n^{1/n})^n} = \frac{x}{1}$ ~~$=x$~~

$= \cancel{x} \cdot 0$

$= 0$

$\therefore$ By cauchy's n^{th} root test Σu_n converges if $x<1$ & diverges if $x>1$

If $x=1$ then the series become $\Sigma \frac{1}{n^n}$

Here $u_n = \frac{1}{n^n} > 0 \; \forall n \in Z^+$

Take $V_n = \frac{1}{n^n}$

Then $\lim_{n\to\infty} \frac{u_n}{V_n} = \frac{1}{\cancel{n^n}} \times \cancel{n^n} = 1 \neq 0$

$\therefore$ By limit comparision test Σu_n & ΣV_n both converges & diverges together

$\Sigma V_n = \sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^n$ is ~~an auxitary series with p≥1~~ geometric series with $r = \frac{1}{n} > 1$

~~$\therefore \Sigma V_n$ converges~~

~~$\therefore \Sigma u_n$ converges~~

$\therefore \Sigma V_n$ diverges

$\therefore \Sigma u_n$ diverges

$\therefore \Sigma \frac{x^n}{n^n}$, $x > 0$ converges if $x < 1$ & diverges if $x \geq 1$

3 Sol: let $u_n = \left(\frac{n+1}{2n+5}\right)^n > 0 \; \forall n \in z^+$

$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} \left[\left(\frac{n+1}{2n+5}\right)^n\right]^{1/n} = \lim_{n\to\infty} \frac{n+1}{2n+5}$

$= \lim_{n\to\infty} \frac{\cancel{n}[1+1/n]}{\cancel{n}[2+5/n]}$

$= \lim_{n\to\infty} \frac{1+1/n}{2+5/n}$

$= \frac{1}{2} < 1$

$\therefore$ By cauchy's n^{th} root test $\Sigma\left(\frac{n+1}{2n+5}\right)^n$ is convergent

4 Sol: let $u_n = \frac{n^{n^2}}{(n+1)^{n^2}} > 0 \; \forall n \in z^+$

$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} \left[\frac{n^{n^2}}{(n+1)^{n^2}}\right]^{1/n} = \lim_{n\to\infty} \frac{n^n}{(n+1)^n}$

$= \lim_{n\to\infty} \frac{\cancel{n^n}}{\cancel{n^n}\left(1+\frac{1}{n}\right)^n}$

$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1$

$\therefore$ By cauchy's n^{th} root test Σu_n is convergent

5 Sol: let $u_n = n \cdot e^{-n^2} > 0 \; \forall n \in z^+$

$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} \left[n \cdot e^{-n^2}\right]^{1/n} = \lim_{n\to\infty} n^{1/n} \cdot e^{-n^2/n}$

$= \lim_{n\to\infty} n^{1/n} \cdot e^{-n} = 0 < 1$

$\therefore$ By cauchy's n^{th} root test Σu_n is convergent

* D'Alembert's Ratio Test:

Statement: If Σu_n is a series of positive terms such that

$\lim \frac{u_{n+1}}{u_n} = l$ then

(i) Σu_n converges if $l<1$

(ii) Σu_n diverges if $l>1$ and

(iii) The test fails to decide the nature of the series if $l=1$

Proof: let Σu_n be a series of positive terms such that $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = l$

$u_n > 0 \ \forall \ n \in z^+ \Rightarrow \frac{u_{n+1}}{u_n} > 0 \ \forall \ n \in z^+ \Rightarrow l \geq 0$

$\lim \frac{u_{n+1}}{u_n} = l \Rightarrow$ for $\varepsilon > 0 \ \exists \ m \in z^+ \ni \left|\frac{u_{n+1}}{u_n} - l\right| < \varepsilon \ \forall \ n \geq m$

$\Rightarrow -\varepsilon < \frac{u_{n+1}}{u_n} - l < \varepsilon$

$\Rightarrow l-\varepsilon < \frac{u_{n+1}}{u_n} < l+\varepsilon \ \forall \ n \geq m$ — (1)

Put $n = m, m+1, m+2 \dots n-1$ in (1)

$n=m: \ l-\varepsilon < \frac{u_{m+1}}{u_m} < l+\varepsilon$

$n=m+1: \ l-\varepsilon < \frac{u_{m+2}}{u_{m+1}} < l+\varepsilon$

$n=m+2: \ l-\varepsilon < \frac{u_{m+3}}{u_{m+2}} < l+\varepsilon$

$\vdots$

$n=n-1: \ l-\varepsilon < \frac{u_n}{u_{n-1}} < l+\varepsilon$

Multiplying these n-m inequalities

$(l-\varepsilon)^{n-m} < \frac{u_{m+1}}{u_m} \cdot \frac{u_{m+2}}{u_{m+1}} \cdot \frac{u_{m+3}}{u_{m+2}} \cdots \frac{u_n}{u_{n-1}} < (l+\varepsilon)^{n-m}$

$\Rightarrow (l-\varepsilon)^{n-m} < \frac{u_n}{u_m} < (l+\varepsilon)^{n-m}$

$\Rightarrow \frac{u_m}{(l-\varepsilon)^m}(l-\varepsilon)^n < u_n < \frac{u_m}{(l+\varepsilon)^m}(l+\varepsilon)^n \ \forall \ n \geq m$

$\Rightarrow \alpha(l-\varepsilon)^n < u_n < \beta(l+\varepsilon)^n$ — (2) $\forall \ n \geq m$

where $\alpha = \frac{u_m}{(l-\varepsilon)^m}, \beta = \frac{u_m}{(l+\varepsilon)^m}$

case(i): let $l<1$

choose $\varepsilon > 0 \ni K = l+\varepsilon < 1$

From (2) we have

$u_n < \beta(l+\varepsilon)^n \ \forall \ n \geq m$

$\Rightarrow u_n < \beta K^n \ \forall \ n \geq m$

Here ΣK^n is a geometric series with common ratio $K<1$

$\therefore \Sigma K^n$ is convergent

Hence by comparision test Σu_n is convergent

case(ii): let $l>1$

choose $\varepsilon > 0 \ni K = l-\varepsilon > 1$

From (2) we have

$\alpha(l-\varepsilon)^n < u_n \ \forall \ n \geq m$

$\alpha K^n < u_n \ \forall \ n \geq m \Rightarrow u_n > \alpha K^n \ \forall \ n \geq m$

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Here ΣK^n is a geometric series with common ratio

$K > 1$

$\therefore \Sigma K^n$ divergers.

Hence by comparision test ΣU_n is divergent.

Case (iii):- let $l = 1$

consider a series $\Sigma \frac{1}{n}$

Here $u_n = \frac{1}{n}$

$$\lim \frac{u_{n+1}}{u_n} = \lim \frac{1}{n+1} \times \frac{n}{1} = \lim \frac{1}{1+\frac{1}{n}}$$

$$= 1$$

$\Sigma \frac{1}{n}$ is an auxilary series with $P = 1$

$\therefore \Sigma \frac{1}{n}$ is divergent.

Consider $\Sigma \frac{1}{n^2}$

Here $u_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{1}{(n+1)^2} \times \frac{n^2}{1} = \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^2} = 1$$

$\Sigma \frac{1}{n^2}$ is an auxilary series with $P = 2 > 1$

$\therefore \Sigma \frac{1}{n^2}$ is convergent.

Note: When the ratio test fails Raabi's test decides the nature of the series.

Statement of Raabi's test: If ΣU_n is a series of Positive terms such that $\lim_{n\to\infty} n\left(\frac{U_n}{U_{n+1}} - 1\right) = l$

then (i) ΣU_n converges if $l > 1$

(ii) ΣU_n diverges if $l < 1$ &

(iii) The test fails if $l = 1$

* Test for convergence $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$

Sol: Given series is $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$

let $u_0 = 1,\ u_1 = \frac{1}{1!},\ u_2 = \frac{1}{2!},\ u_n = \frac{1}{n!} > 0 \ \forall n \in z^+$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{1}{(n+1)!} \times \frac{n!}{1} = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$$

$\therefore$ By D'Alembert's Ratio test, ΣU_n is convergent

$$\sum_{n=0}^{\infty} \frac{1}{n!} = e$$

* Test for convergence $\Sigma \frac{2^n - 2}{2^n + 1} x^n,\ x > 0$

Sol: let $u_n = \frac{2^n - 2}{2^n + 1} x^n \geq 0 \ \forall n \in z^+$

Now $\lim u_{n+1} = \frac{2^{n+1} - 2}{2^{n+1} + 1} x^{n+1}$

$$\lim \frac{u_{n+1}}{u_n} = \lim \frac{2^{n+1} - 2}{2^{n+1} + 1} x^{n+1} \times \frac{2^n + 1}{2^n - 2} \frac{1}{x^n}$$

$$= \lim \frac{2^{n+1}\left(1 - \frac{2}{2^{n+1}}\right)}{2^{n+1}\left(1 + \frac{1}{2^{n+1}}\right)} \times \frac{2^n\left(1 + \frac{1}{2^n}\right)}{2^n\left(1 - \frac{2}{2^n}\right)} \cdot x \quad \left[\because \lim \frac{1}{2^n} = 0\right]$$

$\therefore$ By ratio test Σu_n converges if $x<1$ & diverges if $x>1$

If $x=1$, the series becomes

$$\Sigma \frac{2^n-2}{2^n+1}$$

Here $u_n = \frac{2^n-2}{2^n+1}$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{2^n\left(1-\frac{2}{2^n}\right)}{2^n\left(1+\frac{1}{2^n}\right)} = \lim_{n\to\infty}\left[\frac{1-\frac{2}{2^n}}{1+\frac{1}{2^n}}\right]$$

$$= 1 \neq 0$$

$\therefore$ By n^{th} term test Σu_n is divergent

Hence $\Sigma \frac{2^n-2}{2^n+1} \cdot x^n$, $(x>0)$ converges if $x<1$

diverges if $x \geq 1$

* Test for convergence $\Sigma \frac{x^n}{x^n+a^n}$ $(x>0, a>0)$

Sol: Given series is $\Sigma \frac{x^n}{x^n+a^n}$

let $u_n = \frac{x^n}{x^n+a^n} > 0 \ \forall \ n \in z^+$

Case(i): let $x<a$. Then $0 < \frac{x}{a} < 1$

$$\Rightarrow \lim_{n\to\infty} \left(\frac{x}{a}\right)^n = 0$$

Now $\lim \frac{u_{n+1}}{u_n} = \lim \frac{x^{n+1}}{x^{n+1}+a^{n+1}} \times \frac{x^n+a^n}{x^n}$

$$= \lim x \cdot \frac{a^n\left(1+\left(\frac{x}{a}\right)^n\right)}{a\, a^{n}\left(1+\left(\frac{x}{a}\right)^{n+1}\right)}$$

$$= \frac{x}{a} < 1$$

$\therefore$ By ratio test Σu_n is convergent

Case (ii): let $x=a$. Then

$$u_n = \frac{a^n}{a^n+a^n} = \frac{a^n}{2a^n} = \frac{1}{2}$$

$\lim u_n = \frac{1}{2} \neq 0$

By n^{th} term test Σu_n is divergent

Case(iii): Let $x>a$ then $0 < \frac{a}{x} < 1$

$$\lim_{n\to\infty} \left(\frac{a}{x}\right)^n = 0$$

$$u_n = \frac{x^n}{x^n+a^n} = \frac{x^n}{x^n\left(1+\left(\frac{a}{x}\right)^n\right)} =$$

$$\lim u_n = \lim \frac{1}{1+\left(\frac{a}{x}\right)^n} = 1 \neq 0$$

$\therefore$ By n^{th} term test Σu_n is divergent

* Test for convergence $\sum_{n=1}^{\infty} \frac{1\cdot3\cdot5 \cdots (2n-1)}{2\cdot4\cdot6 \cdots (2n)} x^{n-1}$ $(x>0)$

Sol: let $u_n = \frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots(2n)} x^{n-1} > 0 \ \forall \ n \in z^+$

$$u_{n+1} = \frac{1\cdot3\cdot5\cdots(2n-1)(2n+1)}{2\cdot4\cdot6\cdots(2n)(2n+2)} x^n$$

$$\lim \frac{u_{n+1}}{u_n} = \lim \frac{2n+1}{2n+2} x = \lim \frac{n\left(2+\frac{1}{n}\right)}{n\left(2+\frac{2}{n}\right)} x$$

$$= \frac{2x}{2} = x$$

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By ratio test $\sum u_n$ converges if $x<1$
diverges if $x>1$

If $x=1$ then ratio test fails

By Raabi's test

$$\lim n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim n\left(\frac{2n+2}{2n+1}-1\right)$$

$$= \lim n\left(\frac{2n+2-2n-1}{2n+1}\right)$$

$$= \lim \frac{n}{2n+1} = \lim \frac{n}{n(2+\frac{1}{n})} = \frac{1}{2} < 1$$

$\therefore$ By Raabi's test $\sum u_n$ is divergent

$\therefore \sum \frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots(2n)} x^{n-1}$ converges if $x<1$
diverges if $x\geq 1$

(1) Test for convergence $\frac{1}{2}+\frac{1\cdot3}{2\cdot5}+\frac{1\cdot3\cdot5}{2\cdot5\cdot8}+\cdots$

(2) Test for convergence $\sum_{n=1}^{\infty}\frac{(n+1)!}{3^n}$

(3) Test for convergence (a) $1+\frac{2^p}{2!}+\frac{3^p}{3!}+\frac{4^p}{4!}+\cdots$ PER

(b) $\sum_{n=1}^{\infty}\frac{n^4}{n!}$

(4) Test for convergence $\sum \frac{1}{1+2+2^2+\cdots+2^{n-1}}$

(5) Test for convergence $\sum_{n=1}^{\infty}\frac{n!}{n^n}$

(6) Test for convergence $1+3x+5x^2+7x^3+\cdots$ $(x>0)$

(7) Test for convergence $2x+\frac{3x^2}{8}+\frac{4x^3}{27}+\cdots$ $(x>0)$

(8) Test for convergence $\frac{1}{2\sqrt{1}}+\frac{x^2}{3\sqrt{2}}+\frac{x^4}{4\sqrt{3}}+\cdots$ $(x>0)$

(9) Test for convergence $\sum \frac{n^2(n+1)^2}{n!}$

(10) Test for convergence $\sum \frac{x^{n-1}}{n^2+1}$ $(x>0)$

1 Sol: Given series $\frac{1}{2}+\frac{1\cdot3}{2\cdot5}+\frac{1\cdot3\cdot5}{2\cdot5\cdot8}+\cdots$

$$u_n = \frac{1\cdot3\cdot5\cdot7\cdots(2n-1)}{2\cdot5\cdot8\cdots(3n-1)} > 0 \ \forall n\in Z^+$$

$$u_{n+1} = \frac{1\cdot3\cdot5\cdot7\cdots(2n-1)(2n+1)}{2\cdot5\cdot8\cdots(3n-1)(3n+2)}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{2n+1}{3n+2}$$

$$= \lim_{n\to\infty}\frac{n(2+\frac{1}{n})}{n(3+\frac{2}{n})} = \frac{2}{3} < 1$$

$\therefore$ By D'Alembert's ratio test $\sum u_n$ is convergent

2 Sol: Given $\sum_{n=1}^{\infty}\frac{(n+1)!}{3^n}$

$$u_n = \frac{(n+1)!}{3^n}$$

$$u_{n+1} = \frac{(n+2)!}{3^{n+1}}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{(n+2)!}{3^{n+1}}\times\frac{3^n}{(n+1)!}$$

$$= \lim_{n\to\infty} \frac{(n+2)\cancel{(n+1)}}{3^n \cdot 3} \times 3^n$$

$$= \lim_{n\to\infty} \frac{n+2}{3}$$

3 Sol. (a) Given Series $1 + \frac{2^p}{2!} + \frac{3^p}{3!} + \cdots$

$$u_n = \frac{n^p}{n!} \quad \& \quad u_{n+1} = \frac{(n+1)^p}{(n+1)!}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)^p}{(n+1)!} \times \frac{n!}{n^p}$$

$$= \lim_{n\to\infty} \frac{(n+1)^p}{n^p} \cdot \frac{n!}{(n+1)!}$$

$$= \lim_{n\to\infty} \frac{\cancel{n^p}\left(1+\frac{1}{n}\right)^p}{\cancel{n^p}} \cdot \frac{1}{n+1}$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^p \cdot \frac{1}{n+1} = \lim_{n\to\infty} \frac{1}{n+1} \cdot \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^p$$

$$= 0 \times 1 = 0$$

$\therefore$ By Ratio test $\sum u_n$ is convergent.

(b) $\sum_{n=1}^{\infty} \frac{n^4}{n!}$

$$u_n = \frac{n^4}{n!}, \quad u_{n+1} = \frac{(n+1)^4}{(n+1)!}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)^4}{(n+1)!} \times \frac{n!}{n^4}$$

$$= \lim_{n\to\infty} \frac{\cancel{n^4}\left(1+\frac{1}{n}\right)^4}{\cancel{n^4}} \times \frac{n!}{(n+1)!}$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^4 \times \frac{1}{(n+1)}$$

$$= 0 \times 1 = 0$$

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$\therefore$ By ratio test $\sum u_n$ is convergent.

4 Sol: Given $\sum \frac{1}{1+2+2^2+\cdots+2^{n-1}}$

$u_n = \frac{1}{2^{n-1}}$ & $u_{n+1} = \frac{1}{2^n}$

$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{1}{2^n} \times 2^{n-1}$

$= \lim_{n\to\infty} \frac{2^n \cdot 2^{-1}}{2^n} = \lim_{n\to\infty} 2^{-1} = \frac{1}{2} < 1$

$\therefore$ By ratio test $\sum u_n$ is convergent.

5 Sol: $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

* Alternating series: A series whose terms are alternatively positive and negative is called an alternating series.

An alternating series is written as $u_1 - u_2 + u_3 - u_4 - \dots = \sum_{n=1}^{\infty} (-1)^{n-1} u_n$

Imp

* Leibnitz's test:-

Statement: If $\{u_n\}$ is a ~~series~~ sequence of positive terms such that

(a) $u_1 \geq u_2 \geq u_3 \geq \dots \geq u_n \geq u_{n+1} \geq \dots$ and

(b) $\lim_{n\to\infty} u_n = 0$

Then the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$ is convergent.

Proof: let $\{u_n\}$ be a sequence of positive terms such that

(a) $u_1 \geq u_2 \geq u_3 \geq \dots \geq u_n \geq u_{n+1} \geq \dots$ and

(b) $\lim_{n\to\infty} u_n = 0$ $\quad (-1)^{2n-1} u_{2n}$

Now we have to prove $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$ is convergent

let $S_n = u_1 - u_2 + u_3 - u_4 + \dots + (-1)^{n-1} u_n$

Now $S_{2n} = u_1 - u_2 + u_3 - u_4 + \dots + u_{2n-1} - u_{2n}$

$S_{2n+2} = u_1 - u_2 + u_3 - u_4 + \dots + u_{2n-1} - u_{2n} + u_{2n+1} - u_{2n+2}$

$S_{2n+2} - S_{2n} = u_{2n+1} - u_{2n+2} > 0$ $[\because \{u_n\}$ is decreasing$]$

$\Rightarrow S_{2n+2} > S_{2n} \quad \forall n \in z^+$

$\therefore \{S_{2n}\}$ is increasing.

Now $S_{2n} = u_1 - u_2 + u_3 - u_4 - \dots + u_{2n-1} - u_{2n}$

$= u_1 - (u_2 - u_3) - (u_4 - u_5) - \dots - (u_{2n-2} - u_{2n-1}) - u_{2n}$

$= u_1 - k$ where $k = (u_2-u_3)+(u_4-u_5)+ \cdots + u_{2n} > 0$

$< u_1$

i.e., $S_{2n} < u_1 \ \forall n \in Z^+$

$\therefore \{S_{2n}\}$ is bounded above with an upper bound u_1

By theorem "An increasing sequence which is bounded above is convergent" the $\{S_{2n}\}$ is convergent.

let $\lim_{n\to\infty} S_{2n} = l$

$S_{2n-1} = u_1 - u_2 + u_3 + u_4 + \cdots + u_{2n-1}$

$S_{2n} = u_1 - u_2 + u_3 - u_4 + \cdots + u_{2n-1} - u_{2n}$

$= S_{2n-1} - u_{2n}$

$\Rightarrow S_{2n-1} = S_{2n} + u_{2n}$

Now $\lim_{n\to\infty} S_{2n-1} = \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} u_{2n}$

$= l + 0 = l$

i.e., the subsequences $\{S_{2n-1}\}$ and $\{S_{2n}\}$ of $\{S_n\}$ converges to the same limit 'l'.

$\therefore$ The sequence $\{S_n\}$ converges.

Hence the series $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$ converges.

* Test the convergence $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

Sol: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$

let $u_n = \frac{1}{n} > 0 \ \forall n \in Z^+$

$u_n - u_{n+1} = \frac{1}{n} - \frac{1}{n+1} = \frac{n+1-n}{n(n+1)} = \frac{1}{n(n+1)} > 0$

$\Rightarrow u_n > u_{n+1} \ \forall \ n \in Z^+$

$\therefore \{u_n\}$ is a decreasing sequence of positive terms.

$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = 0$

$\therefore$ By leibnitz's test, $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ is convergent.

* Examine the convergence of $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)$

Sol: let $u_n = \frac{1}{n}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right) > 0 \ \forall \ n \in Z^+$

$u_{n+1} = \frac{1}{n+1}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}+\frac{1}{n+1}\right)$

Now $u_n - u_{n+1} = \frac{1}{n}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right) - \frac{1}{n+1}\left(1+\frac{1}{2}+\cdots+\frac{1}{n+1}\right)$

$= \left(1+\frac{1}{2}+\cdots\frac{1}{n}\right)\left(\frac{1}{n}-\frac{1}{n+1}\right) - \frac{1}{(n+1)(n+1)}$

$= \left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)\left(\frac{1}{n(n+1)}\right) - \frac{1}{(n+1)(n+1)}$

$= \frac{1}{n(n+1)}\left[1+\frac{1}{2}+\cdots+\frac{1}{n} - \frac{n}{n+1}\right]$

$= \frac{1}{n(n+1)}\left[1+\frac{1}{2}+\cdots+\frac{1}{n} - \left(\frac{1}{n+1}+\frac{1}{n+1}+\cdots+\frac{1}{n+1} \text{(n times)}\right)\right]$

$= \frac{1}{n(n+1)}\left[\left(1-\frac{1}{n+1}\right)+\left(\frac{1}{2}-\frac{1}{n+1}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)\right]$

> 0

i.e., $u_n > u_{n+1} \ \forall \ n \in Z^+$

$\therefore u_n$ is a decreasing sequence ~~with~~ of positive terms

$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right) = 0$ [By cauchy's 1st theorem in sequences]

$\therefore$ By Leibnitz's test $\sum_{n=1}^{\infty}(-1)^{n-1}\cdot\frac{1}{n}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)$ is convergent.

* Absolute and conditional convergence:

The series Σu_n is said to be absolutely convergent if $\Sigma|u_n|$ is convergent. ~~If~~

If Σu_n is convergent and $\Sigma|u_n|$ is divergent then Σu_n is said to be conditionally convergent

* Theorem: If Σu_n converges absolutely then Σu_n converges (or)

An absolute convergent series is always convergent.

Proof: let Σu_n be absolutely convergent

i.e., $\Sigma|u_n|$ is convergent

For each $n \in Z^+$, $-|u_n| \le u_n \le |u_n|$

$$\Rightarrow -|u_n|+|u_n| \le u_n+|u_n| \le |u_n|+|u_n|$$

$$\Rightarrow 0 \le u_n+|u_n| \le 2|u_n|$$

$\because$ $\Sigma|u_n|$ is convergent, $\Sigma 2|u_n|$ is convergent.

Now, by comparision test of the first kind, $\Sigma(u_n+|u_n|)$ is convergent.

$\Sigma(u_n+|u_n|)$, $\Sigma|u_n|$ are convergent

$$\Rightarrow \Sigma(u_n+|u_n|-|u_n|) = \Sigma u_n \text{ is convergent.}$$

Note:
~~Converse~~

Converse of this theorem is not true. i.e., if Σu_n is convergent and $\Sigma|u_n|$ need not be convergent.

For example, consider a series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$.

By Leibnitz's test $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is convergent.

But $\sum_{n=1}^{\infty} \left|\frac{(-1)^{n-1}}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ is an auxilary series with $p=1$.

$\therefore \sum_{n=1}^{\infty} \left|\frac{(-1)^{n-1}}{n}\right|$ is not convergent.

* Examine the convergence and absolute convergence of the series $\sum_{n=1}^{\infty} (-1)^{n+1}(\sqrt{n+1}-\sqrt{n})$

Sol: let $u_n = \sqrt{n+1}-\sqrt{n}$

$$= \frac{\sqrt{n+1}-\sqrt{n} \times \sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$$

$$= \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}} > 0 \quad \forall n \in Z^+$$

$$u_{n+1} = \frac{1}{\sqrt{n+2}+\sqrt{n+1}}$$

$$\sqrt{n+2}+\sqrt{n+1} \geq \sqrt{n+1}+\sqrt{n}$$

$$\Rightarrow \frac{1}{\sqrt{n+2}+\sqrt{n+1}} \leq \frac{1}{\sqrt{n+1}+\sqrt{n}} \Rightarrow u_{n+1} \leq u_n \quad \forall n \in Z^+$$

$\therefore \{u_n\}$ is a decreasing sequence of Positive terms.

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$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{n}\left(\sqrt{1+\frac{1}{n}}+1\right)}$$

$$= 0$$

$\therefore$ By leibneitz's test $\sum_{n=1}^{\infty} (-1)^{n+1}\left(\sqrt{n+1}-\sqrt{n}\right)$ is convergent

$$\sum_{n=1}^{\infty} \left|(-1)^{n+1}\left(\sqrt{n+1}-\sqrt{n}\right)\right|$$

$$= \sum_{n=1}^{\infty} \left(\sqrt{n+1}-\sqrt{n}\right)$$

let $u_n = \sqrt{n+1}-\sqrt{n}$

$$= \frac{\sqrt{n+1}-\sqrt{n} \times \sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$$

$$= \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}} > 0 \ \forall n \in Z^+$$

$$\Rightarrow u_n = \frac{1}{\sqrt{n}\left[\sqrt{1+\frac{1}{n}}+1\right]} > 0 \ \forall n \in Z^+$$

Take $v_n = \frac{1}{\sqrt{n}} > 0 \ \forall n \in Z^+$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{\cancel{\sqrt{n}}\left[\sqrt{1+\frac{1}{n}}+1\right]} \times \cancel{\sqrt{n}}$$

$$= \frac{1}{\sqrt{1+\frac{1}{n}}+1}$$

$$= \frac{1}{2} \neq 0$$

By limit comparision test both converge or diverge together

Here $\sum v_n = \sum \frac{1}{\sqrt{n}} = \sum \frac{1}{n^{1/2}}$ is an auxilary series

$P = \frac{1}{2} < 1$ then it diverges.

$\therefore \sum v_n$ divergent

Then $\sum u_n$ divergent

$\therefore$ By limit comparision test $\sum u_n$ is divergent.

i.e., $\sum_{n=1}^{\infty} (-1)^{n+1} (\sqrt{n+1} - \sqrt{n})$ is convergent test but

$\sum_{n=1}^{\infty} |(-1)^{n+1} (\sqrt{n+1} - \sqrt{n})|$ is divergent

Hence the given series is conditionally convergent.

* Prove that $\sum (-1)^{n-1} \frac{x^n}{n}$ is convergent for $-1 < x \leq 1$.

Sol:- Given series is $\sum (-1)^{n-1} \frac{x^n}{n}$.

case(i): let $x = 1$

Then $\sum (-1)^{n-1} \frac{x^n}{n} = \sum \frac{(-1)^{n-1}}{n}$

let $u_n = \frac{1}{n} > 0 \ \forall n \in Z^+$

$u_n - u_{n+1} = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} > 0$

$\Rightarrow u_n > u_{n+1} \ \forall n \in Z^+$

$\therefore u_n$ is a decreasing sequence of positive terms.

$\lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{1}{n} = 0$

$\therefore$ By Leibnitz's test $\sum (-1)^{n-1} u_n = \sum \frac{(-1)^{n-1}}{n}$ is convergent

Case (ii): Let $-1 < x < 1$. Then $|x| < 1$

Let $u_n = (-1)^{n-1} \frac{x^n}{n}$

Now $|u_n| = \left|\frac{(-1)^{n-1} x^n}{n}\right| = \frac{|x|^n}{n}$

$|u_{n+1}| = \frac{|x|^{n+1}}{n+1}$

Now, $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \frac{\frac{|x|^{n+1}}{n+1}}{\frac{|x|^n}{n}} = \lim_{n\to\infty} \frac{|x|^{\cancel{n+1}}}{n+1} \cdot \cancel{n}$

$= \lim_{n\to\infty} \frac{|x|}{(1+\frac{1}{n})}$

$= |x| < 1$

$\therefore$ By D'Alembert's ratio test $\sum |u_n|$ is convergent.

$\Rightarrow \sum u_n$ is convergent.

Hence $\sum (-1)^{n-1} \frac{x^n}{n}$ converges for $-1 < x \le 1$

Auxilary series (or) P-Series Test:

Statement: The series $\sum \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$,

PER (i) converges if $p > 1$ &

(ii) diverges if $p \le 1$

Proof: The auxilary series is

$$\sum \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$$

(i) let $p > 1$

let $S_n = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p}$ be the n^{th} partial

$$\sum \frac{1}{n^p}$$

$$\frac{1}{1^p} = 1 \qquad 2^0 \text{ (1 term)}$$

$$\frac{1}{2^p} + \frac{1}{3^p} < \frac{1}{2^p} + \frac{1}{2^p} = \frac{2}{2^p} = \frac{1}{2^{p-1}} \qquad 2^1 \text{ (2 terms)}$$

$$\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} < \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p}$$

$$< \frac{4}{4^p} < \frac{1}{(2^2)^{p-1}} \qquad 2^2 \text{ (4 terms} = 2^2 \text{ te}$$

$$\frac{1}{8^p} + \frac{1}{9^p} + \frac{1}{10^p} + \frac{1}{11^p} + \frac{1}{12^p} + \frac{1}{13^p} + \frac{1}{14^p} + \frac{1}{15^p} \qquad 2^3 \text{ (8 terms} = 2^3 \text{ te}$$

$$< \frac{1}{8^p} + \frac{1}{8^p} + \cdots + \frac{1}{8^p} \text{ (8 terms)}$$

$$< \frac{8}{8^p} < \frac{1}{(2^3)^{p-1}}$$

— — — — — —

— — — —

$$\frac{1}{(2^n)^p} + \frac{1}{(2^n+1)^p} + \frac{1}{(2^n+2)^p} + \cdots + \frac{1}{(2^{n+1}-1)^p}$$

$$< \frac{1}{(2^n)^p} + \frac{1}{(2^n)^p} + \frac{1}{(2^n)^p} + \cdots + \frac{1}{(2^n)^p} \quad [2^n \text{ te}$$

$$< \frac{2^n}{(2^n)^p}$$

$$< \frac{1}{(2^n)^{p-1}}$$

Adding the above inequalities, the ~~first~~ sum of the first $(2^{n+1}-1)$ terms becomes

$$1+\frac{1}{2^p}+\frac{1}{3^p}+\cdots+\frac{1}{(2^n)^p}+\frac{1}{(2^n+1)^p}+\cdots+\frac{1}{(2^{n+1}-1)^p}$$

$$< 1+\frac{1}{2^{p-1}}+\frac{1}{(2^2)^{p-1}}+\frac{1}{(2^3)^{p-1}}+\cdots+\frac{1}{(2^n)^{p-1}}$$

i.e., $S_{2^{n+1}-1} < 1+\frac{1}{2^{p-1}}+\frac{1}{(2^{p-1})^2}+\frac{1}{(2^{p-1})^3}+\cdots\frac{1}{(2^{p-1})^n}$

[This is a geometric series with $a=1, r=\frac{1}{2^{p-1}}$, $(n+1)$ terms]

$$< \frac{1\left(1-\left(\frac{1}{2^{p-1}}\right)^{n+1}\right)}{1-\frac{1}{2^{p-1}}}$$

$$< \frac{1}{1-\frac{1}{2^{p-1}}} \quad \forall n\in Z^+$$

For each $n\in Z^+$, $2^{n+1}-1 > 2^n > n$

$\therefore S_n < S_{2^n} < S_{2^{n+1}-1} < \frac{1}{1-\frac{1}{2^{p-1}}}$

i.e., $S_n < \frac{1}{1-\frac{1}{2^{p-1}}} \ \forall n\in Z^+$

$\therefore \{S_n\}$ is bounded above

Since $\frac{1}{n^p} > 0 \ \forall n\in Z^+$, $\{S_n\}$ is increasing

Sol: let $u_n = \frac{x^n}{n^n} > 0 \; \forall \, n \in z^+$

$$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty}\left[\frac{x^n}{n^n}\right]^{1/n} = \lim_{n\to\infty}\frac{x}{(n^{1/n})^n} = \frac{\cancel{x}}{1} \cancel{=x}$$

$$= \cancel{x}\cdot 0$$

$$= 0$$

$\therefore$ By cauchy's n^{th} root test Σu_n converges if $x<1$, diverges if $x>1$

If $x=1$ then the series become $\Sigma \frac{1}{n^n}$

Here $u_n = \frac{1}{n^n} > 0 \; \forall \, n \in Z^+$

Take $V_n = \frac{1}{n^n}$

Then $\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{\cancel{n^n}} \times \cancel{n^n} = 1 \neq 0$

$\therefore$ By limit comparision test Σu_n & ΣV_n both converges or diverges together.

$\Sigma V_n = \sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^n$ is ~~an auxitary series with p≥1~~ geometric series with $r = \frac{1}{n} > 1$

~~$\therefore \Sigma V_n$ converges~~

~~$\therefore \Sigma u_n$ converges~~

$\therefore \Sigma V_n$ diverges

$\therefore \Sigma u_n$ diverges

$\therefore \Sigma \frac{x^n}{n^n}$, $x>0$ ~~is a~~ converges if $x<1$ & diverges if $x \geq 1$

3 Sol: let $u_n = \left(\frac{n+1}{2n+5}\right)^n > 0 \ \forall n \in z^+$

$$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty}\left[\left(\frac{n+1}{2n+5}\right)^n\right]^{1/n} = \lim_{n\to\infty}\frac{n+1}{2n+5}$$

$$= \lim_{n\to\infty}\frac{\cancel{n}[1+1/n]}{\cancel{n}[2+5/n]}$$

$$= \lim_{n\to\infty}\frac{1+1/n}{2+5/n}$$

$$= \frac{1}{2} < 1$$

$\therefore$ By cauchy's n^{th} root test $\Sigma\left(\frac{n+1}{2n+5}\right)^n$ is convergent

4 Sol: let $u_n = \frac{n^{n^2}}{(n+1)^{n^2}} > 0 \ \forall n \in z^+$

$$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty}\left[\frac{n^{n^2}}{(n+1)^{n^2}}\right]^{1/n} = \lim_{n\to\infty}\frac{n^n}{(n+1)^n}$$

$$= \lim_{n\to\infty}\frac{\cancel{n^n}}{\cancel{n^n}\left(1+\frac{1}{n}\right)^n}$$

$$= \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1$$

$\therefore$ By cauchy's n^{th} root test Σu_n is convergent

5 Sol: let $u_n = n\cdot e^{-n^2} > 0 \ \forall n \in z^+$

$$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty}\left[n\cdot e^{-n^2}\right]^{1/n} = \lim_{n\to\infty} n^{1/n}\cdot e^{-n^2/n}$$

$$= \lim_{n\to\infty} n^{1/n}\cdot e^{-n} = 0 < 1$$

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∴ By cauchy's n^{th} root test Σu_n is conv...

* D'Alembert's Ratio Test:

Statement: If Σu_n is a series of positive terms su...

$\lim \frac{u_{n+1}}{u_n} = l$ then

(i) Σu_n converges if $l<1$

(ii) Σu_n diverges if $l>1$ and

(iii) The test fails to decide the nature of the series ...

$l = 1$

Proof: let Σu_n be a series of positive terms s...

that $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = l$

$u_n > 0 \ \forall\, n \in z^+ \Rightarrow \frac{u_{n+1}}{u_n} > 0 \ \forall\, n \in z^+ \Rightarrow l \geq 0$

$\lim \frac{u_{n+1}}{u_n} = l \Rightarrow$ for $\varepsilon > 0 \ \exists\, m \in z^+ \ni \left| \frac{u_{n+1}}{u_n} - l \right| < \varepsilon \ \forall$ $n \geq$...

$\Rightarrow -\varepsilon < \frac{u_{n+1}}{u_n} - l < \varepsilon$

$\Rightarrow l - \varepsilon < \frac{u_{n+1}}{u_n} < l + \varepsilon \ \forall\, n \geq m$ — (1)

put $n = m, m+1, m+2 \cdots n-1$ in (1)

$n=m: \quad l - \varepsilon < \frac{u_{m+1}}{u_m} < l + \varepsilon$

$n=m+1: \quad l - \varepsilon < \frac{u_{m+2}}{u_{m+1}} < l + \varepsilon$

$n=m+2: \quad l - \varepsilon < \frac{u_{m+3}}{u_{m+2}} < l + \varepsilon$

$\vdots \qquad \vdots$

$n=n-1: \quad l - \varepsilon < \frac{u_n}{u_{n-1}} < l + \varepsilon$

Multiplying these n-m inequalities

$$(l-\varepsilon)^{n-m} < \frac{u_{m+1}}{u_m} \cdot \frac{u_{m+2}}{\cancel{u_{m+1}}} \cdot \frac{\cancel{u_{m+3}}}{\cancel{u_{m+2}}} \cdots \frac{u_n}{\cancel{u_{n-1}}} < (l+\varepsilon)^{n-m}$$

$$\Rightarrow (l-\varepsilon)^{n-m} < \frac{u_n}{u_m} < (l+\varepsilon)^{n-m}$$

$$\Rightarrow \frac{u_m}{(l-\varepsilon)^m}(l-\varepsilon)^n < u_n < \frac{u_m}{(l+\varepsilon)^m} (l+\varepsilon)^n \ \forall\ n \geq m$$

$$\Rightarrow \alpha(l-\varepsilon)^n < u_n < \beta(l+\varepsilon)^n \quad \text{—(2)} \ \forall\ n \geq m$$

where $\alpha = \frac{u_m}{(l-\varepsilon)^m}$, $\beta = \frac{u_m}{(l+\varepsilon)^m}$

case(i): let $l < 1$

choose $\varepsilon > 0 \ni K = l+\varepsilon < 1$

From (2) we have

$u_n < \beta(l+\varepsilon)^n \ \forall\ n \geq m$

$\Rightarrow u_n < \beta K^n \ \forall\ n \geq m$

Here ΣK^n is a geometric series with common ratio $K < 1$

$\therefore \Sigma K^n$ is convergent

Hence by comparision test Σu_n is convergent.

case(ii): let $l > 1$

choose $\varepsilon > 0 \ni K = l-\varepsilon > 1$

From (2) we have

$\alpha(l-\varepsilon)^n < u_n \ \forall\ n \geq m$

$\alpha K^n < u_n \ \forall\ n \geq m \Rightarrow u_n > \alpha K^n \ \forall\ n \geq m$

Here ΣK^n is a geometric series with common ratio $K > 1$

$\therefore \Sigma K^n$ divergers.

Hence by comparision test Σu_n is divergent.

Case (iii):- let $l = 1$

consider a series $\Sigma \frac{1}{n}$

Here $u_n = \frac{1}{n}$

$$\lim \frac{u_{n+1}}{u_n} = \lim \frac{1}{n+1} \times \frac{n}{1} = \lim \frac{1}{1+\frac{1}{n}}$$

$$= 1$$

$\Sigma \frac{1}{n}$ is an auxilary series with $P = 1$

$\therefore \Sigma \frac{1}{n}$ is divergent.

Consider $\Sigma \frac{1}{n^2}$

Here $u_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{1}{(n+1)^2} \times \frac{n^2}{1} = \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^2} = 1$$

$\Sigma \frac{1}{n^2}$ is an auxilary series with $P = 2 > 1$

$\therefore \Sigma \frac{1}{n^2}$ is convergent.

Note: When the ratio test fails Raabi's test decides the nature of the series.

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Statement of Raabi's test: If Σu_n is a series of Positive terms such that $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = l$

then (i) Σu_n converges if $l > 1$

(ii) Σu_n diverges if $l < 1$ &

(iii) The test fails if $l = 1$

* Test for convergence $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$

Sol: Given series is $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$

let $u_0 = 1$, $u_1 = \frac{1}{1!}$, $u_2 = \frac{1}{2!}$, $u_n = \frac{1}{n!} > 0 \ \forall n \in z^+$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{1}{(n+1)!} \times \frac{n!}{1} = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$$

$\therefore$ ~~Σ~~ By D'Alembert's Ratio test, Σu_n is convergent

$$\sum_{n=0}^{\infty} \frac{1}{n!} = e$$

* Test for convergence $\Sigma \frac{2^n - 2}{2^n + 1} x^n$, $x > 0$

Sol: let $u_n = \frac{2^n - 2}{2^n + 1} x^n \geq 0 \ \forall n \in z^+$

Now $\lim u_{n+1} = \frac{2^{n+1} - 2}{2^{n+1} + 1} x^{n+1}$

$$\lim \frac{u_{n+1}}{u_n} = \lim \frac{2^{n+1} - 2}{2^{n+1} + 1} x^{n+1} \times \frac{2^n + 1}{2^n - 2} \frac{1}{x^n}$$

$$= \lim \frac{\cancel{2^{n+1}}\left(1 - \frac{2}{2^{n+1}}\right)}{\cancel{2^{n+1}}\left(1 + \frac{1}{2^{n+1}}\right)} \times \frac{\cancel{2^n}\left(1 + \frac{1}{2^n}\right)}{\cancel{2^n}\left(1 - \frac{2}{2^n}\right)} \cdot x$$

$$= x \qquad \left[\because \lim \frac{1}{2^n} = 0\right]$$

$\therefore$ By ratio test Σu_n converges if $x<1$ &
diverges if $x>1$

If $x=1$, the series becomes

$$\Sigma \frac{2^n-2}{2^n+1}$$

Here $u_n = \dfrac{2^n-2}{2^n+1}$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{\cancel{2^n}\left(1-\frac{2}{2^n}\right)}{\cancel{2^n}\left(1+\frac{1}{2^n}\right)} = \lim_{n\to\infty}\left[\frac{1-\frac{2}{2^n}}{1+\frac{1}{2^n}}\right]$$

$$= 1 \neq 0$$

$\therefore$ By n^{th} term test Σu_n is divergent

Hence $\Sigma \dfrac{2^n-2}{2^n+1}\cdot x^n$, $(x>0)$ converges if $x<1$
diverges if $x\geq 1$

* Test for convergence $\Sigma \dfrac{x^n}{x^n+a^n}$ $(x>0, a>0)$

Sol: Given series is $\Sigma \dfrac{x^n}{x^n+a^n}$

let $u_n = \dfrac{x^n}{x^n+a^n} > 0 \ \forall\ n \in Z^+$

case(i): let $x<a$. Then $0<\dfrac{x}{a}<1$

$$\Rightarrow \lim_{n\to\infty}\left(\frac{x}{a}\right)^n = 0$$

Now $\lim \dfrac{u_{n+1}}{u_n} = \lim \dfrac{\cancel{x^{n}}\,x}{x^{n+1}+a^{n+1}} \times \dfrac{x^n+a^n}{\cancel{x^n}}$

$$= \lim x \frac{\cancel{a^n}\left(1+\left(\frac{x}{a}\right)^n\right)}{a\,\cancel{a^{n}}\left(1+\left(\frac{x}{a}\right)^{n+1}\right)}$$

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$= \frac{x}{a} < 1$

$\therefore$ By ratio test $\sum u_n$ is convergent

case (ii): let $x = a$. Then

$$u_n = \frac{a^n}{a^n + a^n} = \frac{\cancel{a^n}}{2\cancel{a^n}} = \frac{1}{2}$$

$$\lim u_n = \frac{1}{2} \neq 0$$

By n^{th} term test $\sum u_n$ is divergent

case (iii): let $x > a$ then $0 < \frac{a}{x} < 1$

$$\lim_{n\to\infty} \left(\frac{a}{x}\right)^n = 0$$

$$u_n = \frac{x^n}{x^n + a^n} = \frac{\cancel{x^n}}{\cancel{x^n}\left(1+\left(\frac{a}{x}\right)^n\right)}$$

$$\lim u_n = \lim \frac{1}{1+\left(\frac{a}{x}\right)^n} = 1 \neq 0$$

$\therefore$ By n^{th} term test $\sum u_n$ is divergent

* Test for convergence $\sum_{n=1}^{\infty} \frac{1.3.5.---(2n-1)}{2.4.6.---(2n)} x^{n-1} \; (x > 0)$

Sol: let $u_n = \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2\cdot 4\cdot 6 \cdots (2n)} x^{n-1} > 0 \; \forall \; n \in Z^+$

$$u_{n+1} = \frac{1\cdot 3\cdot 5 \cdots (2n-1)(2n+1)}{2\cdot 4\cdot 6 \cdots (2n)(2n+2)} x^n$$

$$\lim \frac{u_{n+1}}{u_n} = \lim \frac{2n+1}{2n+2} \cdot x = \lim \frac{\cancel{n}(2+\frac{1}{n})}{\cancel{n}(2+\frac{2}{n})} x$$

$$= \frac{2x}{2} = x$$

By ratio test, Σu_n converges if $x<1$

diverges if $x>1$

If $x=1$ then ratio test fails

By Rabbi's test

$$\lim n\left(\frac{u_n}{u_{n+1}}-1\right)=\lim n\left(\frac{2n+2}{2n+1}-1\right)$$

$$=\lim n\left(\frac{\cancel{2n}+2-\cancel{2n}-1}{2n+1}\right)$$

$$=\lim \frac{n}{2n+1}=\lim \frac{\cancel{n}}{\cancel{n}(2+\frac{1}{n})}=\frac{1}{2}<1$$

$\therefore$ By Raabi's test Σu_n is divergent

$\therefore \Sigma \frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots(2n)}x^{n-1}$ converges if $x<1$

diverges if $x\geq1$

(1) Test for convergence $\frac{1}{2}+\frac{1\cdot3}{2\cdot5}+\frac{1\cdot3\cdot5}{2\cdot5\cdot8}+\cdots$

(2) Test for convergence $\sum_{n=1}^{\infty}\frac{(n+1)!}{3^n}$

(3) Test for convergence (a) $1+\frac{2^p}{2!}+\frac{3^p}{3!}+\frac{4^p}{4!}+\cdots$ PER

(b) $\sum_{n=1}^{\infty}\frac{n^4}{n!}$

(4) Test for convergence $\Sigma\frac{1}{1+2+2^2+\cdots+2^{n-1}}$

(5) Test for convergence $\sum_{n=1}^{\infty}\frac{n!}{n^n}$

(6) Test for convergence $1+3x+5x^2+7x^3+\cdots\ (x>0)$

(7) Test for convergence $2x+\frac{3x^2}{8}+\frac{4x^3}{27}+\cdots\ (x>0)$

(8) Test for convergence $\frac{1}{2\sqrt{1}}+\frac{x^2}{3\sqrt{2}}+\frac{x^4}{4\sqrt{3}}+\cdots\ (x>0)$

(9) Test for convergence $\sum \frac{n^2(n+1)^2}{n!}$

(10) Test for convergence $\sum \frac{x^{n-1}}{n^2+1}\ (x>0)$

1Sol: Given series $\frac{1}{2}+\frac{1\cdot 3}{2\cdot 5}+\frac{1\cdot 3\cdot 5}{2\cdot 5\cdot 8}+\cdots$

$$u_n = \frac{1\cdot 3\cdot 5\cdot 7\cdots(2n-1)}{2\cdot 5\cdot 8\cdots(3n-1)} > 0 \ \forall\, n\in Z^+$$

$$u_{n+1} = \frac{1\cdot 3\cdot 5\cdot 7\cdots(2n-1)(2n+1)}{2\cdot 5\cdot 8\cdots(3n-1)(3n+2)}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{2n+1}{3n+2}$$

$$= \lim_{n\to\infty}\frac{\cancel{n}(2+\frac{1}{n})}{\cancel{n}(3+\frac{2}{n})} = \frac{2}{3} < 1$$

$\therefore$ By D'Alembert's ratio test $\sum u_n$ is convergent

2Sol: Given $\sum_{n=1}^{\infty}\frac{(n+1)!}{3^n}$

$$u_n = \frac{(n+1)!}{3^n}$$

$$u_{n+1} = \frac{(n+2)!}{3^{n+1}}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{(n+2)!}{3^{n+1}}\times\frac{3^n}{(n+1)!}$$

$$= \lim_{n\to\infty} \frac{(n+2)\cancel{(n+1)}}{3^n \cdot 3} \times 3^n$$

$$= \lim_{n\to\infty} \frac{n+2}{3}$$

3Sol (a) Given Series $1 + \frac{2^p}{2!} + \frac{3^p}{3!} + \cdots$

$$u_n = \frac{n^p}{n!} \;\&\; u_{n+1} = \frac{(n+1)^p}{(n+1)!}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)^p}{(n+1)!} \times \frac{n!}{n^p}$$

$$= \lim_{n\to\infty} \frac{(n+1)^p}{n^p} \cdot \frac{n!}{(n+1)!}$$

$$= \lim_{n\to\infty} \frac{\cancel{n^p}\left(1+\frac{1}{n}\right)^p}{\cancel{n^p}} \cdot \frac{1}{n+1}$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^p \cdot \frac{1}{n+1} = \lim_{n\to\infty} \frac{1}{n+1} \cdot \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^p$$

$$= 0 \times 1 = 0$$

$\therefore$ By Ratio test $\sum u_n$ is Convergent.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^4}{n!}$

$$u_n = \frac{n^4}{n!}, \quad u_{n+1} = \frac{(n+1)^4}{(n+1)!}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)^4}{(n+1)!} \times \frac{n!}{n^4}$$

$$= \lim_{n\to\infty} \frac{\cancel{n^4}\left(1+\frac{1}{n}\right)^4}{\cancel{n^4}} \times \frac{n!}{(n+1)!}$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^4 \times \frac{1}{(n+1)}$$

$$= 0 \times 1 = 0$$

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$\therefore$ By ratio test $\sum u_n$ is convergent.

Sol: Given $\sum \frac{1}{1+2+2^2+\cdots+2^{n-1}}$

$u_n = \frac{1}{2^{n-1}}$ & $u_{n+1} = \frac{1}{2^n}$

$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{1}{2^n} \times 2^{n-1} = \lim_{n\to\infty} \frac{2^n \cdot 2^{-1}}{2^n} = \lim_{n\to\infty} 2^{-1} = \frac{1}{2} < 1$

$\therefore$ By ratio test $\sum u_n$ is convergent.

Sol: $\sum_{n=1}^{\infty} \frac{n!}{n^n}$